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Understanding atmospheric dynamics using rearrangements of functions

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Abstract

Applications of the mathematical theory of rearrangements of functions to atmosphere and ocean dynamics are discussed. This technique allows treatment of the most important nonlinearity, that due to advection, and full exploitation of Lagrangian conservation quantities, such as potential vorticity. It also allows treatment of non-smooth flows, such as vortex sheets. We illustrate the techniques by using them to prove that a state of geostrophic and hydrostatic balance minimises the energy subject to conservation of absolute momentum and potential temperature, and show that a steady state of a semigeostrophic flow can be characterised as a stationary point of the energy subject to rearrangements of the potential vorticity. We apply them to nonlinear stability theory for semi-geostrophic shear flows, allowing extension of and new insights on published results.

Newton Institute preprint NI96011

1 Introduction

The description of weather systems in the atmosphere has always drawn a great deal on a Lagrangian description of the flow. Obvious examples are the description of synoptic developments in terms of air masses, the parcel theory of convection, and the description of the dynamics of precipitation systems in terms of conveyor belts, (e.g. Browning and Roberts(1994)). The importance of Lagrangian concepts in dynamical understanding has been highlighted in recent years by the extensive use of potential vorticity diagnostics, (Hoskins et al. (1985)). The power of Lagrangian methods has been demonstrated by contour dynamics algorithms and by methods of analysis based on contour displacements (Dritschel (1988ab)). However, the difficulty of relating Lagrangian particle coordinates to the dynamical fields has restricted the use of Lagrangian theory in understanding the properties of the solutions of the governing equations. A feature of Eulerian formulations is that the advection problem becomes nonlinear unless the advecting velocity is uniform. In general, little can be said about the solution of nonlinear problems, apart from verifying global conservation properties, so that the analysis stops with the linear case. However, the process being described is quite simple, the advected quantity is transported along trajectories with no change in value. This is exploited by Lagrangian numerical methods, and should be susceptible to analysis.

A concept that arises naturally when considering the Lagrangian description of incompressible fluid motion is that of a rearrangement. Let σ represent a quantity, associated

with a fluid, which is conserved following the motion of a fluid particle, such as potential vorticity in adiabatic flow. Assume we are working in a bounded region. As time advances, the fluid particles will, in general, exchange their spatial positions, but each particle retains its original value of σ. Then we can say that, for any given time t, σ considered as a function $\sigma(\mathbf{x}, t)$ is a *rearrangement* of the initial function $\sigma(\mathbf{x}, 0)$, where $\mathbf{x}$ is the typical Eulerian position vector. Alternatively we say that σ is *rearrangement preserved* by the flow. In this paper, we define this concept precisely and then show how some tools available from rearrangement theory can be used in analysis of atmospheric (or oceanic) flows. Further mathematical details and proofs are given by Douglas (1996).

Three practical results are illustrated. The first is the characterisation of minimum energy states subject to conservation of properties following particles. It is shown that it is necessary to consider whether the extreme state can be reached by mixing the fluid, in which case the use of conservation following particles may not be appropriate. The second is the behaviour of the time evolution of a flow subject to (potential) vorticity conservation. Again it is crucial whether the long time evolution approximates mixing, equivalent to the enstrophy cascade of two-dimensional turbulence, or whether this is prohibited. The third is the study of the stability of steady states, following the work of Nycander (1995) and Kushner and Shepherd (1995ab). Applying a standard principle for Hamiltonian systems, such as in Abarbanel and Holm (1987), we show that steady states can be characterised as stationary points of the energy under rearrangements of the potential vorticity. Stable steady states correspond to maxima or minima of the energy under rearrangements, following a principle first stated by Kelvin. We obtain new insights on Kushner and Shepherd's results because we are working with a different class of perturbations. Rearrangement perturbations are consistent with the time dependent dynamics, which conserve potential vorticity, and allow consideration of large, non-smooth perturbations. They thus give global, fully nonlinear, results.

2 Atmospheric models and rearrangements

As shown by Hoskins *et al.* (1985), many balanced models of the atmosphere can be described in terms of the Lagrangian conservation of a potential vorticity q by an equation

$$\frac{Dq}{Dt} = 0 \tag{1}$$

and an 'invertibility principle', which allows all other dynamical variables to be determined from knowledge of q. In particular, it determines the advection velocity appearing in (1). This principle will be derived from a balance constraint applied to the full dynamical equations. We write this in the general symbolic form

$$H(p, \mathbf{u}, \theta) = q \tag{2}$$

H is typically an elliptic operator which may be nonlinear. Boundary conditions are also required. Conditions, e.g. non-negativity, may have to be imposed on q to allow (2) to be solved uniquely. Specific examples are

i) Barotropic vorticity equation

Assume this is to be solved in a region Ω with boundary $\partial\Omega$. (2) takes the form

$$\begin{aligned} \nabla^2\psi &= q - f \text{ on } \Omega \\ \psi &= 0 \text{ on } \partial\Omega \\ (v, -u) &= \nabla\psi \end{aligned} \tag{3}$$

where

$$q = \zeta + f, \tag{4}$$

ζ is the relative vorticity and f the Coriolis parameter. The evolution is then governed by (1). (3) can be solved for any values of ζ. There is no solvability condition.

ii) Quasi-geostrophic theory

Following Hoskins et al (1985), (2) takes the form

$$\nabla_h^2\psi + f_0^2\frac{\partial}{\partial p}\left(N^{-2}\frac{\partial\psi}{\partial p}\right) = q - f \tag{5}$$

with boundary condition

$$f_0\frac{\partial\psi}{\partial p} = -R(\theta - \theta_{\text{ref}}) \tag{6}$$

at $p = 0, p_0$. Here, subscript h represents a derivative at constant p, N is the Brunt-Vaisala frequency, and ψ the geostrophic streamfunction. θ_{ref} is a reference potential temperature.

$$q = f + \mathbf{k}.\nabla_p \times \mathbf{v} + f_0\frac{\partial}{\partial p}\left(\frac{\theta - \theta_{\text{ref}}}{d\theta_{\text{ref}}/dp}\right) \tag{7}$$

q is conserved following the horizontal geostrophic motion. (5) can be solved for ψ on a region $\Omega \times [0, p_0]$ with boundary $\partial\Omega$ given the additional boundary condition $\psi = 0$ on $\partial\Omega$. The remaining fields can then be calculated from the geostrophic relation $\mathbf{v} = \mathbf{k} \times \nabla\psi$ and the linearised hydrostatic relation $f_0\partial\psi/\partial p = -R(\theta - \theta_{\text{ref}})$. Since (5) is linear, there is no solvability condition. In particular, there is no distinction between situations where $(q - f)$ is positive or negative.

iii) Semigeostrophic theory on an f plane

This is most simply expressed in the pseudo-height coordinate of Hoskins (1975). Then (2) takes the form that the Hessian of P

$$\det \begin{pmatrix} \frac{\partial^2 P}{\partial x^2} & \frac{\partial^2 P}{\partial x \partial y} & \frac{\partial^2 P}{\partial x \partial z} \\ \frac{\partial^2 P}{\partial y \partial x} & \frac{\partial^2 P}{\partial y^2} & \frac{\partial^2 P}{\partial y \partial z} \\ \frac{\partial^2 P}{\partial z \partial x} & \frac{\partial^2 P}{\partial z \partial y} & \frac{\partial^2 P}{\partial z^2} \end{pmatrix} = q \tag{8}$$

where $P = \phi + \frac{1}{2}(x^2 + y^2)$, ϕ is the geopotential, and the geostrophic wind and potential temperature are given by $\nabla\phi = (fv_g, -fu_g, g\theta/\theta_0)$. In order to solve (8), define $\rho = \frac{1}{q}$ and $S = \mathbf{x}.\mathbf{X} - P$ where the geostrophic and isentropic coordinates $\mathbf{X}$ are given by

$$\mathbf{X} = (x + v_g/f, y - u_g/f, g\theta/(f^2\theta_0)) \tag{9}$$

Then, following Cullen and Purser(1989),

$$\det \begin{pmatrix} \frac{\partial^2 S}{\partial X^2} & \frac{\partial^2 S}{\partial X \partial Y} & \frac{\partial^2 S}{\partial X \partial Z} \\ \frac{\partial^2 S}{\partial Y \partial X} & \frac{\partial^2 S}{\partial Y^2} & \frac{\partial^2 S}{\partial Y \partial Z} \\ \frac{\partial^2 S}{\partial Z \partial X} & \frac{\partial^2 S}{\partial Z \partial Y} & \frac{\partial^2 S}{\partial Z^2} \end{pmatrix} = \rho \tag{10}$$

can be solved in a convex region Ω in (x, y, z) if the right hand side is non-negative subject to the boundary condition

$$\nabla_{\mathbf{X}} S \in \Omega \tag{11}$$

The form of (10) guarantees that S is a convex function. The situation is now different from quasi-geostrophic theory in that the equations cannot be solved for negative potential vorticity.

The solution of all these models can be described as rearrangements of the initial potential vorticity distribution. Since no new values can be created, or existing values destroyed, there is a significant constraint on the resulting flows. The rearrangement is determined by calculating an advecting velocity field using the invertibility principle. This procedure constrains the velocity field and thus gives information about the type of rearrangement that can be reached from given initial data. These ideas have been exploited mathematically for case (i) by Kato (1967) and case (ii) by Bennett and Kloeden (1981), amongst others. These results are summarised in section 6.

Another type of application is the idea of minimising a function such as the energy with respect to rearrangements. Shutts *et al* (1988) illustrated that insight could be obtained into many processes in the atmosphere by using the ideas of minimising the energy of a fluid, subject to certain conservation laws. The simplest example is the rearrangement of a vertical column of fluid where the buoyancy, B, of each fluid particle is fixed. Thus we specify the initial buoyancy as a function $B_0(z)$, and seek that function B(z) which is a rearrangement of B_0 and minimises the energy $\int_0^h -Bzdz$. The minimum energy state is that with the heaviest fluid at the bottom and a monotonic increase of buoyancy with height. In the dynamical evolution, the energy of this minimising state cannot be converted into kinetic energy. This is expressed by the concept of 'available potential energy', which is the difference between the actual potential energy, and the energy of the minimising resting state. Only this can be converted into kinetic energy. Other examples treated by Shutts *et al* were in two-dimensional cross-sections (x, z), where the potential temperature and the y component of absolute momentum $(v + fx)$ are simultaneously

conserved, since there are no pressure gradients in the y direction in such problems. It is shown that two-dimensional semigeostrophic solutions correspond to a sequence of such minimum energy states, where the time evolution of potential temperature and absolute momentum is prescribed.

Cullen *et al* (1987) showed that solutions of the three dimensional semi-geostrophic equations could be interpreted this way, even in spherical geometry. If (X, Y, Z) are the coordinates defined in (9), the solution is found by predicting (X, Y, Z) on fluid particles using the equations

$$\begin{aligned} \dot{X} &= f(y - Y) \\ \dot{Y} &= f(X - x) \\ \dot{Z} &= 0 \end{aligned} \tag{12}$$

and then finding the location of the particles as the minimum energy rearrangement $\mathbf{X}_0 = \mathbf{X}_0(\mathbf{x})$, where the geostrophic energy is that which is minimised. Thus the solution can be characterised as a sequence of minimum energy states.

A different use of the idea of energy minimisation is contained in Vallis (1992). He shows that geostrophic and hydrostatic balance can be characterised as the minimum energy state of the atmosphere with a given potential vorticity distribution. Roulstone and Sewell (private communication) show that if the extremisation is carried out with respect to a given distribution of potential vorticity on fluid particles, then the resulting state is a steady solution of the primitive equations. The latter is an example of the general Hamiltonian principle for steady states, e.g. Abarbanel and Holm (1987).

3 Basic definitions and results from rearrangement theory

3.1 Rearrangements of scalar valued functions

We begin by giving an intuitive idea of when a function is a rearrangement of another function. Let Ω be a bounded region in $\mathbb{R}^n$, and let $f : \Omega \to \mathbb{R}^d$ be a function. Imagine that Ω is a continuum of infinitesimal particles, so that $n = 2$ or 3 in general. Suppose we exchange the particle positions, with each particle retaining its value of f. This yields a new function $g = g(\mathbf{x})$, say, which we call a *rearrangement* of $f = f(\mathbf{x})$. The idea of rearranging a function can be applied to both scalar and vector valued functions. The latter case is important if there is more than one quantity which is conserved by the flow.

We give a simple example of two scalar functions which are rearrangements of each other. Let $f(x) = x$ for each $x \in [0,1]$, and let g be defined by

$$g(x) = \begin{cases} 1-2x & \text{if } x \in [0,1/2], \\ 2x-1 & \text{if } x \in [1/2,1]. \end{cases} \tag{13}$$

For each $0 < \alpha < 1$,

$$\int_0^1 (f(x)-\alpha)_+ dx = \frac{1}{2}(1-\alpha)^2 = \int_0^1 (g(x)-\alpha)_+ dx \tag{14}$$

where the subscript $+$ denotes the positive part of the function. (14) says that the area of the region which is bounded above by the graph of f and bounded below by the line $y = \alpha$ is the same as the area of the corresponding regions of g. This is illustrated in Figure 1: the total shaded areas are the same. If $\alpha \geq 1$, then $\int_0^1 (f(x)-\alpha)_+ dx = 0 = \int_0^1 (g(x)-\alpha)_+ dx$. Combining the above results, $\int_0^1 (f(x)-\alpha)_+ dx = \int_0^1 (g(x)-\alpha)_+ dx$ for each positive α.

We will use this property to make a precise definition of when f and g are *rearrangements* of each other.

Following Eydeland, Spruck and Turkington (1990), let $f, g : \Omega \to \mathbb{R}$ be two non–negative integrable functions i.e $\int_\Omega f(\mathbf{x})d\mathbf{x} < \infty$, $\int_\Omega g(\mathbf{x})d\mathbf{x} < \infty$. f is a *rearrangement* of g if

$$\int_\Omega (f(\mathbf{x})-\alpha)_+ d\mathbf{x} = \int_\Omega (g(\mathbf{x})-\alpha)_+ d\mathbf{x} \tag{15}$$

for each $\alpha > 0$.

In order to make it easier to extend the concept to vector-valued functions, it is more convenient to work with an alternative definition. This is expressed in terms of the measures of sets on which f and g take specified values. The measure of a set indicates its size. In $\mathbb{R}$ the measure could be length, in $\mathbb{R}^2$, area, in $\mathbb{R}^3$, volume. Returning to the example of Fig.1, consider the sets $\{x \in [0,1] : f(x) \in [1/2,1]\}$ and $\{x \in [0,1] : g(x) \in [1/2,1]\}$. The former set is $[1/2,1]$ and the latter $[0,1/4] \bigcup [3/4,1]$. These two sets have the same Lebesgue measure (or length). This is illustrated in Figure 2. A general statement of this property is contained in the following definition.

Definition Let non–negative $f, g : \Omega \to \mathbb{R}$ be integrable functions, where $\Omega \subset \mathbb{R}^n$ is a bounded region. f and g are rearrangements if

$$\mu(\{\mathbf{x} : f(\mathbf{x}) \in B\}) = \mu(\{\mathbf{x} : g(\mathbf{x}) \in B\}) \tag{16}$$

for each Borel set $B \subset \mathbb{R}$, where μ denotes n–dimensional Lebesgue measure. By restricting our definition to Borel sets of $\mathbb{R}$ (which excludes some pathological subsets of $\mathbb{R}$), we exclude sets whose size cannot be calculated, so that (16) is always well defined.

This definition can be shown to be equivalent to the one given earlier. Another equivalent formulation which appears in the literature (see Burton (1987) for example) states that for f and g as in the above definition, f is a rearrangement of g if

$$\mu(\{\mathbf{x} : f(\mathbf{x}) \geq \alpha\}) = \mu(\{\mathbf{x} : g(\mathbf{x}) \geq \alpha\}) \tag{17}$$

for each positive α.

We now introduce the concept of the *set of rearrangements* of a given function. A function $f = f(x)$ has more than one rearrangement (unless it is constant). Two different rearrangements of the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = x$ are illustrated in Figure 3. We write $R(f_0)$ to denote the set of rearrangements of a prescribed function f_0.

3.2 Rearrangements of vector valued functions

We introduce the concept of rearrangement of vector valued functions by an example. Define a function f on the unit square $[0,1] \times [0,1]$ by

$$f(\mathbf{x} = (x,y)) = \begin{cases} (1,1) & \text{if } 1/2 \leq x \leq 1, 1/2 \leq y \leq 1 \\ (0,0) & \text{otherwise,} \end{cases} \tag{18}$$

and another function g by

$$g(\mathbf{x} = (x,y)) = \begin{cases} (0,0) & \text{if } 1/8 < y < 7/8, \\ (1,1) & \text{if } y \leq 1/8 \text{ or } y \geq 7/8. \end{cases} \tag{19}$$

f and g are illustrated in Figure 4. Consider the set of points where f takes the value $(1,1)$, that is $\{\mathbf{x} \in [0,1]\times[0,1] : f(\mathbf{x}) = (1,1)\}$, and the corresponding set for g, $\{\mathbf{x} \in [0,1]\times[0,1] : g(\mathbf{x}) = (1,1)\}$. These sets are equal to $[1/2,1]\times[1/2,1]$ and $[0,1]\times[0,1/8] \bigcup [0,1]\times[7/8,1]$ respectively, and both have area $1/4$. Similarly the sets $\{\mathbf{x} : f(\mathbf{x}) = (0,0)\}$ and $\{\mathbf{x} : g(\mathbf{x}) = (0,0)\}$ have equal area (of $3/4$). As f and g take no other values, it follows that $\{\mathbf{x} : f(\mathbf{x}) \in B\}$ has the same area, or 2–dimensional Lebesgue measure, as $\{\mathbf{x} : g(\mathbf{x}) \in B\}$ for every Borel set $B \subset \mathbb{R}^2$. The restriction to Borel sets has the same purpose as in the previous section.

We will now use this property to give a formal definition of rearrangements for vector valued functions.

Definition Let Ω be a bounded region in $\mathbb{R}^n$, and let $f, g : \Omega \to \mathbb{R}^d$ be integrable functions. Then f and g are *rearrangements* if

$$\mu(\{\mathbf{x} : f(\mathbf{x}) \in B\}) = \mu(\{\mathbf{x} : g(\mathbf{x}) \in B\}) \tag{20}$$

for every Borel subset of $\mathbb{R}^d$, where μ denotes n–dimensional Lebesgue measure.

This definition is identical to that given in (16) for scalar valued rearrangements, except that d will now be greater than 1. There are several other definitions in the literature. Cullen, Norbury and Purser (1991) gave a different definition which is an extension of the characterisation of scalar valued rearrangements given above in (17). They required that for f and g as above, $\mu(\{\mathbf{x} : f(\mathbf{x}) \geq \mathbf{c}\}) = \mu(\{\mathbf{x} : g(\mathbf{x}) \geq \mathbf{c}\})$ for each $\mathbf{c} \in \mathbb{R}^d$, where the inequalities are calculated component by component. A third definition, used in Brenier (1991), requires that $\int_\Omega F(f)d\mathbf{x} = \int_\Omega F(g)d\mathbf{x}$ for each F in a subclass of the continuous functions from $\mathbb{R}^d$ to $\mathbb{R}$. We can unify these definitions as follows: (see Douglas (1996) for details); each may be reduced to the requirement that $\int_\Omega F(f)d\mathbf{x} = \int_\Omega F(g)d\mathbf{x}$ for each $F : \mathbb{R}^d \to \mathbb{R}$ in a class of functions $\mathcal{C}$. The different definitions above correspond to different choices of $\mathcal{C}$. $\mathcal{C}$ must be a sufficiently large class so that for each $n \in \mathbb{N}$ such that $\int_\Omega |f|^n$ is finite, we have $\int_\Omega |f|^n = \int_\Omega |g|^n$, i.e. all moments are preserved.

A vector valued function $f : \Omega \to I\!R^d$ may be written in terms of real valued components $f_i : \Omega \to I\!R$, where $f = (f_1, ..., f_d)$. If f and g are vector valued functions which are rearrangements then the corresponding components of f and g are rearrangements in the scalar valued sense. However it is important to note that the condition that each component of a vector valued function f is a rearrangement (in the scalar sense) of the corresponding component of a vector valued function g, is not sufficient for f to be a rearrangement of g in the vector valued sense. We give an example. Let f be as in the previous example, then

$$f_1(\mathbf{x}) = f_2(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in [1/2,1] \times [1/2,1], \\ 0 & \text{if } \mathbf{x} \notin [1/2,1] \times [1/2,1]. \end{cases} \tag{21}$$

Define

$$g(\mathbf{x}) = \begin{cases} (1,0) & \text{if } \mathbf{x} \in [1/2,1] \times [1/2,1], \\ (0,1) & \text{if } \mathbf{x} \in [0,1/2] \times [1/2,1], \\ (0,0) & \text{otherwise.} \end{cases} \tag{22}$$

Then $g_1 = f_1$ and

$$g_2(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in [0,1/2] \times [1/2,1], \\ 0 & \text{if } \mathbf{x} \notin [0,1/2] \times [1/2,1]. \end{cases} \tag{23}$$

It is easily seen that $f_1 \in R(g_1)$, and $f_2 \in R(g_2)$. However, $f \notin R(g)$, because f takes the value (1,1) on a region of area $1/4$, while g never takes this value. Consequently in general we cannot apply scalar valued rearrangement results to the components of vector valued functions and hope to obtain results about vector valued rearrangements. Indeed, the theory of rearrangements of vector valued functions is less rich than that of rearrangements of scalar valued functions. We will see an example of this in the next section.

3.3 Rearrangements and trajectory mappings

When using rearrangement theory to study the properties of an evolving flow we consider the trajectory mapping that transforms initial positions $\mathbf{x}(\mathbf{a}, 0)$ of fluid particles $\mathbf{a}$ into positions $\mathbf{x}(\mathbf{a}, t)$ at a later time. We use the symbol s for this mapping. Any conserved particle property q is thus transported by this mapping, so that $q(\mathbf{x}(\mathbf{a}, t), t) = q(\mathbf{x}(\mathbf{a}, 0), 0)$. We write this more concisely as $q(t) = q \circ s$. If the fluid is incompressible it follows that, for any t, $q(\mathbf{x}, t)$ is a rearrangement of the initial data $q(\mathbf{x}, 0)$. This is because for any given subset U of the fluid at time t, for each particle in U we can find the original position of the particles at time zero and the set of these points has the same size as U. It follows that the trajectory mapping is a measure preserving mapping, the definition of which we give below.

Definition A *measure preserving mapping* from Ω to itself is a mapping $s : \Omega \to \Omega$ such that for each (measurable) set $U \subset \Omega$, $\mu(\{x : s(x) \in U\}) = \mu(U)$. We must again restrict our definition to sets where we can measure the size of the set - this excludes some pathological choices of U. Halmos (1950) shows that this is equivalent to requiring that for every integrable function f,

$$\int_\Omega f \circ s d\mu = \int_\Omega f d\mu. \tag{24}$$

If a measure preserving mapping s is one to one, and maps measurable sets to measurable sets, then the inverse of s exists and is also a measure preserving mapping. Such an

s is called a *measure preserving transformation*. It can be shown that measure preserving mappings and transformations preserve (scalar or vector valued) rearrangements in the sense that if $f \in R(g)$ and s is a measure preserving mapping, then $f \circ s \in R(g \circ s)$. If s is a measure preserving transformation, then the reverse implication holds. Furthermore, $f \circ s \in R(f)$, and in particular, for a quantity σ which remains constant on fluid particles following the motion of the flow, $\sigma(t) \in R(\sigma(0))$.

Measure preserving mappings are also closely related to *canonical transformations* which are used in Hamiltonian descriptions of the flow. Sewell and Roulstone (1993) give a comparative study of several different definitions of canonical transformation, one of which is

Definition Given a transformation between two pairs of real scalars (x_1, x_2) and (X_1, X_2), with Jacobian

$$\det\left(\frac{\partial(X_1, X_2)}{\partial(x_1, x_2)}\right), \tag{25}$$

the transformation is *canonical* over a subspace of $\mathbb{R}^2$ if the Jacobian is equal to 1 over that subspace.

This definition requires the transformation to be differentiable. The definition of measure preserving mappings given above provides a natural extension to cases where the transformation is non-differentiable. Write $\mathbf{X} = (X_1, X_2)$ and $\mathbf{x} = (x_1, x_2)$. A co-ordinate transformation $\mathbf{X} = \mathbf{X}(\mathbf{x})$ could be considered *canonical* in a generalised sense for example if $\mathbf{X}$ is a measure preserving mapping which preserves important analytic structures such as the form of Hamilton's equations. Sewell and Roulstone review the qualitative properties of such transformations and applications in their paper.

4 Maximising or minimising functions with respect to rearrangements

As discussed in section 2, we are often interested in maximising or minimising functionals such as energy with respect to the set of rearrangements of a prescribed function. The most powerful results are available when there are inequalities which show that a special class of rearrangement achieves the minimisation or maximisation. In this section we review some important rearrangement inequalities; more on this topic can be found in Alvino, Lions and Trombetti (1989) and Burton (1987).

The first such special class are monotone rearrangements. Consider scalar rearrangements first. Let f, g be as in the first example of the section 3, that is

$$f(x) = x, \text{if } x \in [0,1], g(x) = \begin{cases} 1-2x & \text{if } x \in [0,1/2], \\ 2x-1 & \text{if } x \in [1/2,1]. \end{cases} \tag{26}$$

We have previously seen that $f \in R(g)$. f is an increasing function, and it can be proved that it is the only increasing function which is a rearrangement of g. More generally, it can be shown that for an integrable function f_0 defined on $[0,1]$, there is a unique element of $R(f_0)$ which is an increasing function. We call this element the *increasing rearrangement* of f_0, and denote it f_0^*. Increasing rearrangements are used in the following inequality. For scalar valued integrable functions f, g defined on the unit interval, we have

$$\int_0^1 fg \leq \int_0^1 f^* g^* \tag{27}$$

where f^*, g^* are the increasing rearrangements of f and g respectively. Indeed, if g has no level sets of non-zero length,

$$\int_0^1 fg^* < \int_0^1 f^* g^* \tag{28}$$

for every rearrangement f of f_0 which is not equal to f_0^*.

Any other rearrangement may be obtained from the increasing rearrangement by composition with a suitable measure preserving mapping. Ryff (1970) proved that, for $f \in L^1[0,1]$ i.e. for an integrable f defined on $[0,1]$, there exists a measure preserving mapping $s : [0,1] \to [0,1]$ such that $f = f^* \circ s$. Note that it is not always possible to get from one rearrangement to another rearrangement by a measure preserving mapping- for f and g as in the example above, no measure preserving mapping $s : [0,1] \to [0,1]$ such that $f = g \circ s$ exists.

There is an analogous concept for vector valued rearrangements. For $d \geq 2$, $\mathbb{R}^d$ is not well–ordered, therefore the concept of increasing does not make sense for vector valued functions. An increasing scalar valued function is the derivative of a convex function. Replacing derivative with gradient, this is a property which is well defined for vector valued functions, therefore we seek a rearrangement of a vector valued function equal to the gradient of a convex function. Let f be as in the vector valued examples of the previous section, that is $f : [0,1]^2 \to \mathbb{R}^2$ defined by

$$f(\mathbf{x}) = \begin{cases} (1,1) & \text{if } \mathbf{x} \in [1/2,1] \times [1/2,1], \\ (0,0) & \text{if } \mathbf{x} \notin [1/2,1] \times [1/2,1]. \end{cases} \tag{29}$$

Define $f^* : [0,1]^2 \to I\!R^2$ by

$$f^*(\mathbf{x} = (x,y)) = \begin{cases} (1,1) & \text{if } x+y \geq 1 - \frac{1}{\sqrt{2}}, \\ (0,0) & \text{otherwise.} \end{cases} \tag{30}$$

f^* is easily seen to be a rearrangement of f, and moreover $f^* = \nabla\Psi$, where $\Psi : [0,1]^2 \to I\!R$ is a convex function defined by

$$\Psi(\mathbf{x} = (x,y)) = \begin{cases} x + y - \left(1 - \frac{1}{\sqrt{2}}\right) & \text{if } x+y \geq 1 - \frac{1}{\sqrt{2}}, \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

It can be shown that f^* is the unique rearrangement of f equal to the gradient of a convex function. More generally, if $\Omega \subset I\!R^d$ is a bounded closed convex set with smooth boundary, and $f_0 : \Omega \to I\!R^d$ is integrable, Brenier (1991) proved that there exists a unique $f_0^* \in R(f_0)$, with f_0^* equal to the gradient of a convex function. f_0^* is called the *monotone rearrangement* of f_0.

If f_0 satisfies a non-degeneracy condition, essentially that f_0 does not take a uniform value on any set of positive measure, Brenier (1991) has shown the existence of a unique measure preserving mapping $s : \Omega \to \Omega$ such that $f_0 = f_0^* \circ s$, and furthermore, s is the unique measure preserving mapping that maximises $\int_\Omega f_0(\mathbf{x}).s(\mathbf{x})d\mu(\mathbf{x})$. We can use this result to prove a rearrangement inequality. Let $f \in R(f_0)$ with $f \neq f_0^*$. Then there exists a measure preserving mapping s such that $f = f_0^* \circ s$. s is a measure preserving mapping, therefore we have

$$\int_\Omega f_0^*(\mathbf{x}).\mathbf{x}d\mu(\mathbf{x}) = \int_\Omega f_0^*(s(\mathbf{x})).s(\mathbf{x})d\mu(\mathbf{x}) \tag{32}$$

Using $f = f_0^* \circ s$ to substitute for f_0 on the right hand side of (32) gives

$$\int_\Omega f_0^*(\mathbf{x}).\mathbf{x}d\mu(\mathbf{x}) = \int_\Omega f(\mathbf{x}).s(\mathbf{x})d\mu(\mathbf{x}) \tag{33}$$

Applying the result of Brenier (1991) stated above, noting that the identity map is a measure preserving mapping from Ω to itself which is different from s, gives

$$\int_\Omega f_0^*(\mathbf{x}).\mathbf{x}d\mu(\mathbf{x}) > \int_\Omega f(\mathbf{x}).\mathbf{x}d\mu(\mathbf{x}) \tag{34}$$

Douglas (1996) proved that the above inequality still holds for f_0 which fail the non-degeneracy condition. However, Brenier (1991) shows that there is no inequality analogous to (27) for the monotone rearrangement of a vector valued function. This is an example where the theory of vector valued rearrangements is less rich than that of scalar valued rearrangements.

Another similar class of rearrangements are the (Schwarz) symmetric decreasing rearrangements, used by Nycander (1995). They are a special case of the equimeasurable decreasing rearrangements defined in Crowe, Zweibel and Rosenbloom (1986). For a function $q = q(\mathbf{r})$ of several variables, the symmetrised function is a decreasing function of $r = |\mathbf{r}|$. Formally

Definition Let $D \subset I\!R^3$, and let $q : D \to I\!R$ be an integrable function. For $\alpha \in I\!R$ define $D_\alpha = \{\mathbf{r} \in D : q(\mathbf{r}) \geq \alpha\}$, and let D_α^* be the corresponding symmetrised domain i.e. the sphere with the same volume as D_α. Then, for any given $r \in D^*$, the symmetrically

decreasing rearrangement of q is given by

$$q^*(\mathbf{r}) = \sup\{\alpha : \mathbf{r} \in D_\alpha^*\} \tag{35}$$

Inequality (27) extends to symmetric decreasing rearrangements of functions (defined on bounded domains).

We now illustrate the use of these results. As discussed in section 2, the solution of the semigeostrophic equations at any time can be considered as a minimum energy state, subject to simultaneous rearrangements of absolute momentum and potential temperature. The mathematical result on which this depends requires that the energy of the fluid can be minimised by rearranging a vector-valued function $\mathbf{X}$. The components of $\mathbf{X}$ correspond to geostrophic and isentropic coordinates. The first result available of this type is the one stated in Cullen and Purser (1989):

Theorem 1. Let Ω be a connected convex region in real space with a smooth boundary. If the state of the fluid (at some fixed time) is described by values $\mathbf{X}_0 = (X_0, Y_0, Z_0)$ which are known on fluid particles, and the energy is defined by

$$E(\mathbf{X}) = \frac{1}{2}\int_\Omega X^2 + x^2 + Y^2 + y^2 d\mathbf{x} - \int_\Omega \mathbf{X}.\mathbf{x} d\mathbf{x} \tag{36}$$

for $\mathbf{X} \in R(\mathbf{X}_0)$, where $\mathbf{x} = (x, y, z) \in \Omega$, then the fluid can be rearranged into a state $\mathbf{X}_0^* = \mathbf{X}_0^*(\mathbf{x})$ where
i) $E(\mathbf{X}_0^*) < E(\mathbf{X})$ for any rearrangement $\mathbf{X}$ other than $\mathbf{X}_0^*$ itself.
ii) $\mathbf{X}_0^* = \nabla\psi$ for some convex function ψ.

The solution can also be characterised in terms of the value ρ of an inverse potential vorticity $\frac{\partial \mathbf{x}}{\partial \mathbf{X}}$. Given $\rho = \rho(\mathbf{X})$ and provided that $\int \rho d\mathbf{X} = \mu(\Omega)$ (the total volume of Ω), the theorem allows the construction of a unique convex function $S(\mathbf{X})$ with $\nabla S = \mathbf{x}$ and $\frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \rho$, where ρ is a rearrangement of the given values.

We outline the proof of the theorem using rearrangement inequalities (a different approach was sketched by Cullen and Purser). The detailed proof is given in Douglas (1996). The first integral is conserved under vector valued rearrangements, therefore it is sufficient to show that $\int_\Omega \mathbf{X}.\mathbf{x} d\mu(\mathbf{x}) < \int_\Omega \mathbf{X}_0^*.\mathbf{x} d\mu(\mathbf{x})$ for every $\mathbf{X} \in R(\mathbf{X}_0)$ with $\mathbf{X} \neq \mathbf{X}_0^*$. This follows by (33), even when there is a positive volume of fluid with a uniform value of $\mathbf{X}$.

Physically, fluid with uniform values of $\mathbf{X}$ can be rearranged without changing the energy, so that we can expect non-uniqueness of the minimising rearrangement in terms of the particle positions. However the values $\mathbf{X}(\mathbf{x})$ are still unique.

A similar principle is that maximisation or minimisation of kinetic energy over a family of flows whose vorticities (or a quantity closely related to vorticity) are rearrangements of one another, yields a stable steady flow. This idea had its origins in the work of Kelvin (1910): the modern formulation using rearrangements is due to Benjamin (1976) (in the context of three dimensional vortex rings). We discuss two recent papers which exploited this strategy. Nycander (1995) proved the existence of stable stationary vortex patches in background shear flows with uniform vorticity by maximising energy with respect to rearrangements of the vortex patch. Using an inequality like (27) and another rearrangement inequality he showed that the energy associated with the symmetric decreasing rearrangement (as defined in (35)) of a function is greater than or equal to that associated with the function itself. Burton (1987) studied variational problems for vortex rings in an ideal fluid. He used an inequality similar to (27) in the proof that a functional, representing

kinetic energy and momentum, is maximised relative to the set of rearrangements of a given function, where the given function is a "vorticity–like" quantity. He used another rearrangement inequality to show that a maximiser yields an associated stream function which satisfies the equation of a steady flow.

5 Limits of sequences of rearrangements

Rearrangement inequalities can only be used to study minimisation or maximisation of integrals in special cases. We also need methods which can be used more widely. A standard approach would be to construct a minimising or maximising sequence of rearrangements, and show that this sequence, or a subsequence, has a limit which is also a rearrangement and which achieves the required extreme value. This is the approach to energy minimisation used in Cullen and Purser (1989). However such a limit may not exist. A particular difficulty is that the set of rearrangements of a given function includes arbitrarily fine-grained rearrangements. We must then decide whether to include the limit state in which the fluid is allowed to mix. A similar problem arises in studying the long time behaviour of a conserved quantity like potential vorticity which is rearrangement preserved. Fine grained rearrangements of potential vorticity and tracer concentration can develop, for instance at the stratospheric vortex edge. Therefore we wish to review some properties of sets of rearrangements.

The most useful set of functions to work with for meteorological data is the set of square integrable functions $L^2(\Omega)$. Functions in this space have a finite 'energy', but can have arbitrarily small scale oscillations. This is typical, for instance, of atmospheric wind fields. Formally, $f \in L^2(\Omega)$ if $\int_\Omega f^2 d\mu < \infty$, and we can define a norm on $L^2(\Omega)$ by $||f||_2 = \{\int_\Omega f^2 d\mu\}^{\frac{1}{2}}$.

We now recall some standard definitions, which will enable us to say whether a sequence of functions converges to a limit. For more detail see Friedman (1982), sections 3.2, 4.10, 4.14.

Definitions A sequence $\{f_n\} \subset L^2(\Omega)$ converges strongly to $f \in L^2(\Omega)$ if $||f_n - f||_2 \to 0$ as $n \to \infty$. $S \subset L^2(\Omega)$ is *strongly closed* if for every sequence $\{f_n\}$ in S which converges strongly to a $f \in L^2(\Omega)$, say, then the limit function f is actually in S.

A set $S \subset L^2(\Omega)$ is *strongly compact* if every sequence in S has a subsequence which converges strongly to an element of S. A strongly compact set is strongly closed.

It is useful to have another notion of convergence and compactness which can "ignore" small scale oscillations, and thus be more realistically applied to meteorological data. We give an example. For each $n \in \mathbb{N}$, let $f_n \in L^2([0, 2\pi])$ be defined by $f_n(x) = \sin nx$. The sequence $\{f_n\}$ does not converge in $L^2([0, 2\pi])$, nor does any subsequence. However the f_n approach the constant function with value 0 in an averaged sense, as $\int_0^{2\pi} f_n g \to 0$ as $n \to \infty$ for any $g \in L^2(\Omega)$. We say that f_n *converges weakly* to 0. We give the formal definition.

Definition For all positive integers n, let $f_n \in L^2(\Omega)$, and further suppose that $f \in L^2(\Omega)$. We say that f_n *converges weakly* to f if $\int_\Omega f_n g \to \int_\Omega fg$ as $n \to \infty$ for every $g \in L^2(\Omega)$.

Strong convergence implies weak convergence, but not vice versa.

A set $S \subset L^2(\Omega)$ is *weakly (sequentially) compact* if every sequence in S has a subsequence converging weakly to an element of S. For any sequence in S which has a weak limit, the weak limit belongs to S. Strong compactness implies weak (sequential) compactness.

The set of rearrangements of a non-negative function $f_0 \in L^2(\Omega)$ is (strongly) closed, but not strongly compact or weakly compact unless f_0 is a constant function. For example, there are sequences of rearrangements which converge weakly to limits which are not

rearrangements of the original function (from which it follows that the set is not weakly compact). Let $f_0 : [0,1] \to I\!R$ be defined by

$$f_0(x) = \begin{cases} 0 & \text{if } x \in [0,1/2], \\ 1 & \text{if } x \in [1/2,1]. \end{cases} \tag{37}$$

Define, for $n \in I\!N$,

$$f_n(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0 & \text{if } x \in (m/n,(2m+1)/2n], \\ 1 & \text{if } x \in ((2m+1)/2n,(m+1)/n]. \end{cases} \tag{38}$$

where $m = 0,1,...,n-1$. The functions f_3 and f_8 are illustrated in Figure 5. For each $n \in I\!N$, f_n is equal to zero on a set of length $1/2$, and equal to 1 on a set of length $1/2$, therefore f_n is a rearrangement of f_0 for each $n \in I\!N$. However, given any $g \in L^2(0,1)$, it may be shown that $\int_\Omega f_n g d\mu \to 1/2 \int_\Omega g d\mu$ as $n \to \infty$, that is f_n converges weakly to the constant function with value $1/2$, which is not a rearrangement of f_0.

This behaviour can make it difficult to maximise or minimise a functional with respect to rearrangements. It is quite possible that there will be a sequence of rearrangements giving progressively less energy, for instance, but the limit function will not be a rearrangement. To overcome this difficulty, we work with the weak closure of the set of rearrangements of a given function, the smallest weakly compact set that contains the set of rearrangements. This set contains all the weak limits of sequences of rearrangements, and may therefore, for instance, allow us to construct the functions that maximise or minimise energy. Given a non–negative function $f_0 \in L^2(\Omega)$, this set is equal to the closed convex hull of the set of rearrangements. Douglas (1994) gave the following characterisation

$$\begin{aligned} \overline{\text{conv}R(f_0)} \;=\; \{f \text{ non}-\text{negative} \mid & \int_\Omega (f-\alpha)_+ d\mu \le \int_\Omega (f_0-\alpha)_+ d\mu \\ & \text{for each } \alpha > 0, \int_\Omega f d\mu = \int_\Omega f_0 d\mu.\} \end{aligned} \tag{39}$$

where the + subscript denotes taking the positive part of the function. If we choose f_0 as in the previous example, that is $f_0 : [0,1] \to I\!R$ is defined by

$$f_0(x) = \begin{cases} 0 & \text{if } x \in [0,1/2], \\ 1 & \text{if } x \in [1/2,1], \end{cases} \tag{40}$$

it can be shown that any integrable function $\varphi : [0,1] \to I\!R$ satisfying $0 \le \varphi(x) \le 1$ for each $x \in [0,1]$, and $\int_0^1 \varphi d\mu = 1/2$, belongs to $\overline{\text{conv } R(f_0)}$. This illustrates that $\overline{\text{conv } R(f_0)}$ may be a large class of functions, in particular it includes the constant value $1/2$ which is certainly not a rearrangement of f_0. In general, all rearrangements are included, as are functions derived by smoothing a rearrangement, if the smoothing preserves the value of the integral, and does not introduce new extreme values. Thus the limit of a sequence of fine grained rearrangements will be a smoothed 'average' function. Physically, including these limit functions can be thought of as allowing for a small but finite viscosity or conductivity.

We illustrate these concepts by considering two energy minimisation problems discussed in section 2. We use the simple one-dimensional examples of Shutts *et al* (1988). Suppose we minimise

$$\frac{1}{2}\int_0^1 v_g^2 dx = \frac{1}{2}\int_0^1 (x - X)^2 dx \tag{41}$$

over $X \in R(f_0)$, where f_0 is as in (37). $\int_0^1 X^2 dx$ has the same value for each $X \in R(f_0)$, therefore minimising (41) is equivalent to maximising $\int_0^1 xX\,dx$ over $X \in R(f_0)$. This integral is maximised by taking $X = f_0^* = f_0$, where f_0^* denotes the increasing rearrangement of f_0. This choice yields a minimum energy of $1/24$ for the integral (41). However if we minimise (41) with respect to X in the weak closure of the set of rearrangements of f_0, the function $X(x) = x$ belongs to this set, therefore we can attain a zero minimum energy state. Note that $X(x) = x$ is the weak limit of a sequence of rearrangements. Therefore a lower energy can be achieved by minimising relative to the weak closure of the set of rearrangements, rather than by minimising relative to the set of rearrangements itself.

Now consider minimising the energy of a vertical column of fluid. Physically, the solution is a rearrangement with the potential temperature monotonically increasing with height, corresponding to the inequality quoted in section 4. The energy can be written, using the pseudo-height coordinate of section 2(iii), as

$$\int -\frac{g\theta z}{\theta_0} \tag{42}$$

Minimising (42) is equivalent to maximising $\int zZ$, where Z is the isentropic coordinate $\frac{g\theta}{\theta_0}$. Suppose that we maximise $\int_0^1 zZ$ over $Z \in R(f_0)$, where f_0 is as in (37). The maximum is attained by choosing $Z = f_0^* = f_0$, with $\int_0^1 z f_0 = 3/8$. If we maximise $\int_0^1 zZ$ over Z belonging to the weak closure of the set of rearrangements of f_0, which includes, for example, the constant function $1/2$, no larger value can be obtained. Therefore we cannot achieve a lower energy by maximising relative to the weak closure of the set of rearrangements in this example.

The difference between these examples follows from the form of the energy functionals. In the case of (41), $\int X^2 dx$ is conserved under rearrangements of X, but is not necessarily conserved when the weak limit is taken. For example, suppose we have a sequence $\{f_n\}$ of rearrangements of f_0 which converges weakly to the identity function id. For each n, $\int_0^1 f_n^2 = 1/2$, but $\int_0^1 id^2 = 1/3$. On the other hand, if $\{X_n\}$ converges weakly to X, then $\int xX_n dx \to \int xX_n dx$ as $n \to \infty$.

In the physical case, we have to consider whether terms in functionals we are extremising are robust against smoothing. The distinction is reflected in observations of the atmosphere. A situation where the potential temperature is layered in the vertical is quite common, and reflects that a lower energy cannot be reached by mixing the fluid. The horizontal equivalent of layering of absolute momentum $(v + fx)$ is not as normally observed, since a lower energy can be reached by mixing the fluid to create more uniform values of v.

6 Applications of rearrangement theory in atmospheric dynamics

6.1 Long time evolution of the barotropic vorticity equation

It is possible that increasingly fine-grained solutions will be developed from smooth initial data through time evolution, for instance the 'enstrophy cascade' described by Vallis (1992). As discussed in section 5, the set of rearrangements is not compact, therefore there exist sequences of rearrangements such that no subsequence converges to a rearrangement. One way of overcoming this difficulty is to work with the weak closure of the set of rearrangements, which we defined in the previous section. We examine this idea in more detail in the next two paragraphs. An alternative strategy is to identify a "reachable" compact subset of the set of rearrangements - rearrangements that can actually be reached in the time evolution. One way to do the latter is to bound the gradients of the vorticity or potential vorticity in terms of the initial gradients. In the case of the barotropic vorticity equation (4,3), the vorticity gradients obey the equation

$$\frac{D}{Dt}\nabla(\zeta+f)+\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\end{pmatrix}\nabla(\zeta+f)=0 \tag{43}$$

This can be written in terms of the streamfunction ψ as

$$\frac{D}{Dt}\nabla(\zeta+f)+\begin{pmatrix}-\frac{\partial^2\psi}{\partial x\partial y} & \frac{\partial^2\psi}{\partial x^2}\\ -\frac{\partial^2\psi}{\partial y^2} & \frac{\partial^2\psi}{\partial y\partial x}\end{pmatrix}\nabla(\zeta+f)=0 \tag{44}$$

These equations can be used to estimate the rate of increase of vorticity gradients. The derivatives of the streamfunction can be bounded because the vorticity is bounded by its initial values and by using the boundary condition specified in (3). This is the basis of the results of Kato (1967) and Bennett and Kloeden (1981). The fact that the subset of rearrangements of $\zeta+f$ with bounded gradients is compact makes the proof possible. The size of the bound is very large and grows in time because the most pessimistic possible assumption has to be used. As stated by Bennett and Kloeden, the magnitude of the deformation field has to be assumed everywhere equal to its maximum value, the field everywhere aligned with the direction of the vorticity gradient, and the magnitude of the vorticity gradient everywhere equal to its maximum.

Another method is to embed the set of rearrangements in a larger set, the weak closure of the set of rearrangements. In section 5 it was shown that this set includes all smoothings of the initial vorticity distribution, as well as the rearrangements themselves. This set is weakly compact. There is no control on the vorticity gradients, so the solution is 'generalised', and can include 'fractal' vorticity distributions. This allows us to consider arbitrarily fine-grained solutions resulting from an enstrophy cascade, and the inclusion of the smooth limit functions means that we can also represent the effect of a small but undefined viscosity.

Once this is done, standard methods can be used to show that a solution exists. The technique is as follows. Given an initial distribution of (potential) vorticity q_0 on a region Ω, use the above discussion to identify a set $E(q_0)$, a 'reachable' subset of the weak closure of the set of rearrangements of q_0, which contains all rearrangements of q_0 which can be

reached by evolution of the initial data according to equation (1), together with all the limit points of this set. The flow geometry ensures that this set is (strongly) compact and convex. We seek a solution over an arbitrary time interval $[0,T]$. Make a first guess to this solution by setting q at each time t to be an arbitrary reachable rearrangement $q_0(t)$ of the initial data. By construction, this guess is within $E(q_0)$. At each time, solve the equivalent of (2) to give a velocity field $u_0(t)$:

$$H(u_0) = q_0 \tag{45}$$

Use this velocity field to transport the initial data, according to equation (1). The result is a second 'guess' $q_1(t)$ to the solution. q_1 is a rearrangement of q_0, contained in the compact and convex set $E(q_0)$. The mapping of the first guess q_0 to q_1 is shown to be a continuous function from $E(q_0)$ to $E(q_0)$. It is then possible to use a standard mathematical theorem to show that this mapping has a fixed point q_f. By construction, this is a solution of (1) and (2).

These results do not give any reason why the vorticity distribution should not gradually cascade into fine grained chaos. Such a cascade is the basis of standard two-dimensional turbulence theory, and is characterised by an energy spectrum obeying a k^{-3} power law. To make any further progress, for instance to explain coherent structures, more information about the flow geometry would have to be used to restrict the possible rearrangements.

6.2 Nonlinear stability of semi-geostrophic shear flows

We first define the problem of semi-geostrophic flow in a periodic channel. In the next subsection we derive the conditions for such flows to be steady. In the last subsection we seek candidates for stable steady states using some specific examples of shear flows.

6.2.1 Semi-geostrophic flow in a periodic channel

Consider purely horizontal flow in a periodic channel with period $2L$ between rigid walls $y = \pm D$. The evolution equations (12) take the form

$$\begin{aligned} \dot{X} &= f(\frac{\partial S}{\partial Y} - Y) \\ \dot{Y} &= X - f(\frac{\partial S}{\partial X}) \end{aligned} \tag{46}$$

S is determined from (10), and the boundary conditions (11) become

$$\frac{\partial S}{\partial X}(X,Y) = \frac{\partial S}{\partial X}(X+2L,Y) : \; -D \leq \frac{\partial S}{\partial Y} \leq D \tag{47}$$

The problem can be stated as an evolution equation for the inverse potential vorticity $\rho = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$ to be solved in (X,Y) space. The domain then becomes $(-L,L)\times(-\infty,\infty)$. The evolution equation is

$$\frac{D\rho}{Dt} = 0 \tag{48}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \dot{X}\frac{\partial}{\partial X} + \dot{Y}\frac{\partial}{\partial X} \tag{49}$$

with $(\dot{X}, \dot{Y})$ given by (46). (10) and (47) can be solved for S given any ρ satisfying

$$\begin{aligned} \rho(X+L,Y) &= \rho(X-L,Y) \\ \int_{-\infty}^{\infty}\int_{-L}^{L} \rho dX dY &= 4LD \end{aligned} \tag{50}$$

6.2.2 Characterisation of steady flows

Kushner and Shepherd (1995ab) studied the stability of shear flows and general steady flows governed by the three-dimensional version of these equations. They found a number of difficulties, particularly that the form of the lateral boundary conditions made it impossible to satisfy the Charney-Stern theorem for shear flows, and that the perturbation energy is non-quadratic in the potential vorticity perturbation. They therefore had to use a 'pseudo-energy' instead, and were only able to obtain small amplitude results.

We, instead, restrict the perturbations to rearrangements of ρ. These are the equivalent for the semi-geostrophic problem of the rearrangements of vorticity which can be used to study the barotropic vorticity equation. The latter are often referred to as isovortical perturbations. The extension of this analysis to three dimensions would additionally restrict the rearrangements to Z surfaces. We impose the additional condition that the mean Y over particles cannot be changed. This is implied by the periodic boundary conditions, which mean, using the notation of (8), (9), that $\int_{-L}^{L} v_g dx = \int_{-L}^{L} \frac{\partial \phi}{\partial x} = 0$. Following the general result of Abarbanel and Holm (1987), steady states should correspond to stationary points of the energy under rearrangements of ρ. According to Kelvin's principle, stable steady states should correspond to maxima or minima of the energy under these rearrangements. The energy for the problem (46) can be derived from (36) as

$$\begin{aligned} E = &\quad \tfrac{1}{2}\int_{-D}^{D}\int_{-L}^{L}\left((X-x)^2+(Y-y)^2 dxdy\right) \\ = &\quad \tfrac{1}{2}\int_{-\infty}^{\infty}\int_{-L}^{L}\left((X-x)^2+(Y-y)^2\right)\rho dX dY \end{aligned} \tag{51}$$

Given $\rho = \rho_0(X,Y)$ satisfying (50), seek the conditions under which $\delta E = 0$ for rearrangement perturbations of ρ, so that $\rho + \delta\rho \in R(\rho_0)$. These can be generated as in (24) by keeping ρ fixed on particles in $\mathbf{X}$ space and perturbing $\mathbf{X}$. The perturbation $\delta\mathbf{X}$ must be non-divergent, so can be written as $(-\frac{\partial \alpha}{\partial Y}, \frac{\partial \alpha}{\partial X})$ for an arbitrary function $\alpha(X,Y)$. The restriction that the mean Y cannot be changed is enforced by applying the periodicity condition $\alpha(X-L,Y) = \alpha(X+L,Y)$ to α. The extension of this to three dimensions would restrict the rearrangements to Z surfaces, so that $\delta Z = 0$. This would mean we had similar results to the two-dimensional case, but applied on isentropic surfaces, as in Kushner and Shepherd. We then have

$$\delta E = \int_{-\infty}^{\infty}\int_{-L}^{L}((X-x)\delta X + (Y-y)\delta Y - X\delta x - Y\delta y)\rho dX dY \tag{52}$$

where the integration is taken over particles, so that there is no $\delta\rho$, and we have used the invariance of $\int_{-D}^{D}\int_{-L}^{L} x^2+y^2 dxdy$. The results of section 4 show that

$$\int_{-\infty}^{\infty}\int_{-L}^{L}(-X\delta x - Y\delta y)\rho dX dY = 0 \tag{53}$$

because this is the variation of $\mathbf{x}$ with $\mathbf{X}$ fixed considered in Theorem 1. Substituting the definitions of Ψ and $\delta\mathbf{X}$ gives

$$\int_{-\infty}^{\infty}\int_{-L}^{L}(-\frac{\partial\Psi}{\partial X}\frac{\partial\alpha}{\partial Y} + \frac{\partial\Psi}{\partial Y}\frac{\partial\alpha}{\partial X})\rho dX dY = 0 \tag{54}$$

so

$$\int_{-\infty}^{\infty}\int_{-L}^{L}(\nabla.(\alpha\rho\frac{\partial\Psi}{\partial Y}, -\alpha\rho\frac{\partial\Psi}{\partial X}) - \alpha(-\frac{\partial\Psi}{\partial X}\frac{\partial\rho}{\partial Y} + \frac{\partial\Psi}{\partial Y}\frac{\partial\rho}{\partial X}))dX dY = 0 \tag{55}$$

Assuming that ρ vanishes at a sufficiently large $|Y|$ and requiring (55) to hold for arbitrary α gives

$$-\frac{\partial\Psi}{\partial X}\frac{\partial\rho}{\partial Y} + \frac{\partial\Psi}{\partial Y}\frac{\partial\rho}{\partial X} = 0 \tag{56}$$

This condition is precisely that for the flow to be steady as we can see from (48) and (49).

6.3 Characterisation of stable steady states

Following Kelvin's principle, we expect to characterise stable steady states by maximising or minimising the energy under rearrangements of ρ. We construct a series of examples to illustrate the results of doing this. By a simple analogue of (35), any $\rho(X,Y)$ satisfying (50) with compact support, so that ρ is zero for $|Y| > Y_0$ for some Y_0, can be rearranged into a distribution corresponding to a steady shear flow parallel to the X axis with ρ increasing in Y:

$$\begin{aligned} \rho_0(X,Y) &= g_0(Y) \\ \int_{-\infty}^{\infty} g_0 dY &= 2D \\ \frac{dg_0(Y)}{dY} &\geq 0 \end{aligned} \tag{57}$$

We then seek to extremise E for $\rho \in R(\rho_0)$ subject to the additional condition that the mean Y is fixed:

$$\int_{-\infty}^{\infty} Y\rho dY = \int_{-\infty}^{\infty} Y g_0(Y) dY \tag{58}$$

It is first clear that the maximum energy attainable under these conditions is infinite. Generate a rearrangement by a displacement $\alpha = A\sin(\pi kX/L)$, implying $\delta X = 0, \delta Y = \frac{A\pi k}{L}\cos\left(\frac{\pi kX}{L}\right)$. Since $|y| < D$ for all particles, (51) shows that $E \to \infty$ as $A \to \infty$. It is therefore only meaningful to seek minimum energy states. Note that the maximum energy state may be well defined for the barotropic vorticity equation or quasi-geostrophic equations, because the displacements α of vorticity have to be within the domain, so that $\alpha = 0$ on the domain boundaries, (see Burton and McLeod (1991), Theorem 3.1). The removal of this restriction in the semi-geostrophic case leads to the boundary contributions

to the pseudo-momentum that caused the difficulties discussed by Kushner and Shepherd.

We are therefore interested only in minimum energy states. In order to find likely candidates, we consider some specific examples. First consider the sequence of distributions

$$\rho_n(X,Y) = \begin{cases} 2f_n(\frac{X+L}{2L}) & \text{if } -D \le Y \le D, \\ 0 & \text{otherwise} \end{cases} \tag{59}$$

where f_n is as defined in (37) and (38). The solution $\mathbf{x} = \mathbf{x}(\mathbf{X})$ is required to satisfy

$$\begin{aligned} \frac{\partial y}{\partial Y} &= 1 \\ \frac{\partial x}{\partial X} &= \begin{cases} 0 & \text{if } X = -L, \\ 0 & \text{if } X \in ((2m-n)L/n, (2m+1-n)L/n], \\ 2 & \text{if } X \in [(2m+1-n)L/n, L(2m+2-n)/n], \end{cases} \end{aligned}$$

where $m = 0, 1, ..., n-1$. The energy associated with this solution is $2L^3D/3n^2$. As n tends to infinity, the solution tends weakly to $\rho = 1 : -D \le Y \le D$ so that $\mathbf{X} = \mathbf{x}$. The energy in the limit case is zero.

In this example, the minimum energy is attained by making ρ as close to 1 as possible by mixing values to obtain a mean of 1. This is because $\rho = 1$ corresponds to the identity map $x = X, y = Y$ which has zero energy. In general, we expect that the only candidates for minimum energy states among distributions of the form (57) will be those with $g_0(Y) \le 1$ everywhere, because no mixing can give an increased range of values of Y for which $g(Y) = 1$. Since ρ and g are inverse potential vorticities, this condition excludes values of potential vorticity less than 1, which correspond to anticyclonic relative vorticity. This agrees with Kushner and Shepherd's result that there were no stable shear flows with anticyclonic shear. This argument would also show that no steady states with anticyclonic relative vorticity were stable in a limited domain under semi-geostrophic dynamics. They could be stable in doubly periodic flows, because there is then no region of $\rho = 0$ in the (X,Y) plane to mix with the non-zero values.

We now impose the condition $g_0(Y) \le 1$ in (57), and minimise the energy with respect to distributions of ρ of the form $\rho(X,Y) = g(Y)$, where $g \in R(g_0)$. The energy given by this distribution is

$$2L \int_{-\infty}^{\infty} \frac{1}{2} \left(\int_{-\infty}^{Y} g(Y')dY' - Y \right)^2 g(Y)dY \tag{60}$$

$\int Yg(Y)$ is fixed by (58) and $\int g(Y)dY$ is fixed by (57). We give the results of an explicit calculation for the three cases

$$g_0(Y)/4D = 0 \text{ except } \begin{cases} 1-\eta & \text{where } \eta = Y - 1/6, 0 \le \eta \le 1 \\ \eta & \text{where } \eta = Y + 1/6, 0 \le \eta \le 1 \\ 2Y & \text{if } 0 \le Y \le 1/2,\ 2-2Y \text{ if } 1/2 \le Y \le 1. \end{cases} \tag{61}$$

These are illustrated in Fig.7. The different definitions of η are chosen so that the three distributions have the same mean y, in order to satisfy the constraint (58). The energy associated with the three, setting $y = y_0$ at $Y = -\infty$ in each case, is $y_0^2/4 - y_0/8 + 13/720$ for the first two choices, and $y_0^2/4 - y_0/8 + 11/192$ for the third. This is what would be

expected from the Charney-Stern theorem. The first two represent respectively monotonically decreasing and increasing profiles of ρ. The overall symmetry of the problem means that these have equal energy. The third is a symmetrical distribution of ρ, but has a vanishing vorticity gradient. The first two represent stable steady states, as no lower energy can be reached, despite the violation of the Charney-Stern conditions by the lateral boundary conditions (see Kushner and Shepherd (1995a). This is in agreement with the results of Kushner and Shepherd (1995b), but is a stronger result since there is no amplitude restriction.

This suggests that, in general, stable steady flows in the semi-geostrophic case correspond to monotonically increasing or decreasing ρ distributions which are independent of X and have $\rho \leq 1$. We now need to check that the result survives mixing of the profile. As in section 5, we can construct a sequence of rearrangements of the profile $g_0(Y)$ defined in (57) that converges to a uniform value of g determined by (58). In the example (61), the value would be $1/2$. The energy associated with this is $y_0^2/4 - y_0/8 + 1/48$. This is greater than the value for the monotonic profiles of (61), suggesting that these two states retain their stability even when all weak limits of rearrangements are considered. It remains to be proved that the same holds for general profiles of ρ other than the linear one assumed in (61).

7 Discussion

We have discussed a number of applications of rearrangement theory to meteorology. We have shown that in models which can be described in terms of potential vorticity and an invertibility principle that useful nonlinear and large amplitude information can be obtained about the flow evolution. The use of variational characterisations of stable steady states as energy extrema allows extensions of nonlinear stability results to large amplitude and non-smooth flows. We have shown that the most powerful results occur in cases where we can use a rearrangement inequality to extremise a functional over all possible rearrangements. Examples quoted used monotone or symmetrising rearrangements. It is important to identify more such situations and exploit them.

Other possible applications are to numerical methods and data assimilation. It is clearly desirable to satisfy the rearrangement property in a numerical solution of the evolution equation for potential vorticity, and to preserve it as well as possible in formulations where potential vorticity is not used as a model variable. This should form part of the assessment of competing numerical methods. In data assimilation, it may be appealing to try and work with displacements or rearrangements of fields in the analysis procedure, as suggested by Hoffman *et al* (1995), rather than with Eulerian perturbations. This could include minimisation over rearrangements as part of a general variational framework.

Acknowledgements

The authors wish to thank Professor M. Sewell for his detailed review of this manuscript. The work was supported by the Isaac Newton Institute through a Programme on the Mathematics of Atmosphere and Ocean Dynamics. The second author's research was supported by EPSRC Research Fellow Grant no. 21409 MTA S08.

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8 Figure Captions

Figure 1. Graphs of $f(x) = x, x \in [0,1]$ and

$$g(x) = \begin{cases} 1-2x & \text{if } x \in [0,1/2], \\ 2x-1 & \text{if } x \in [1/2,1]. \end{cases} \tag{62}$$

illustrating the reason why f is a rearrangement of g.

Figure 2. Graphs of the inverse images of f and g, with $f = f(x)$, $g = g(x)$ defined as in Fig.1. The sets $f^{-1}[1/2,1]$ and $g^{-1}[1/2,1]$ are illustrated.
Figure 3. Graphs of two different rearrangements of $f(x) = x, x \in [0,1]$. The shaded areas are the same.
Figure 4. Values of the vector valued functions $f = f(x,y)$ and $g = g(x,y)$ defined on $[0,1] \times [0,1]$ by (18) and (19) respectively.
Figure 5. The rearrangements $f_3 = f_3(x)$ and $f_8 = f_8(x)$ defined by equation (38) of the function $f_0 = f_0(x)$ defined by equation (37).
Figure 6. The effect on the surface $P = P(x)$ of local concentrations of its curvature.
Figure 7. Three example distributions of the inverse potential vorticity for a semi-geostrophic shear flow.

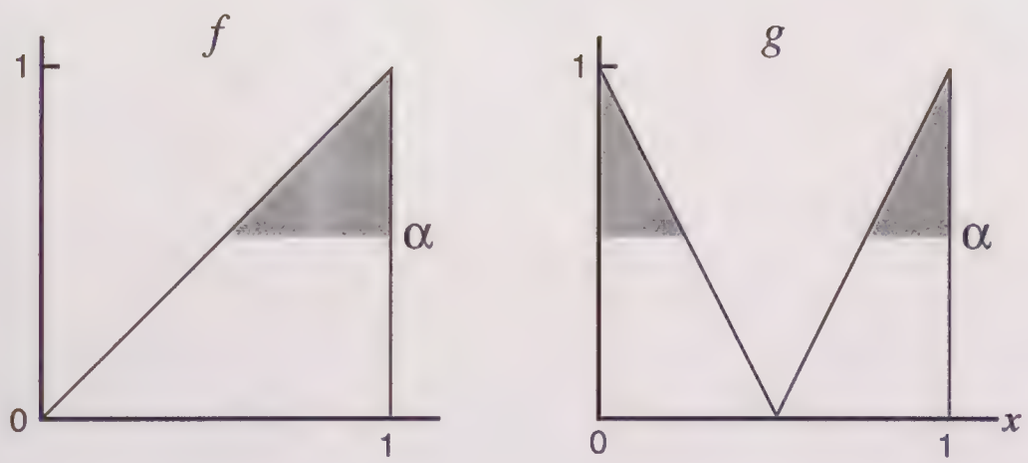
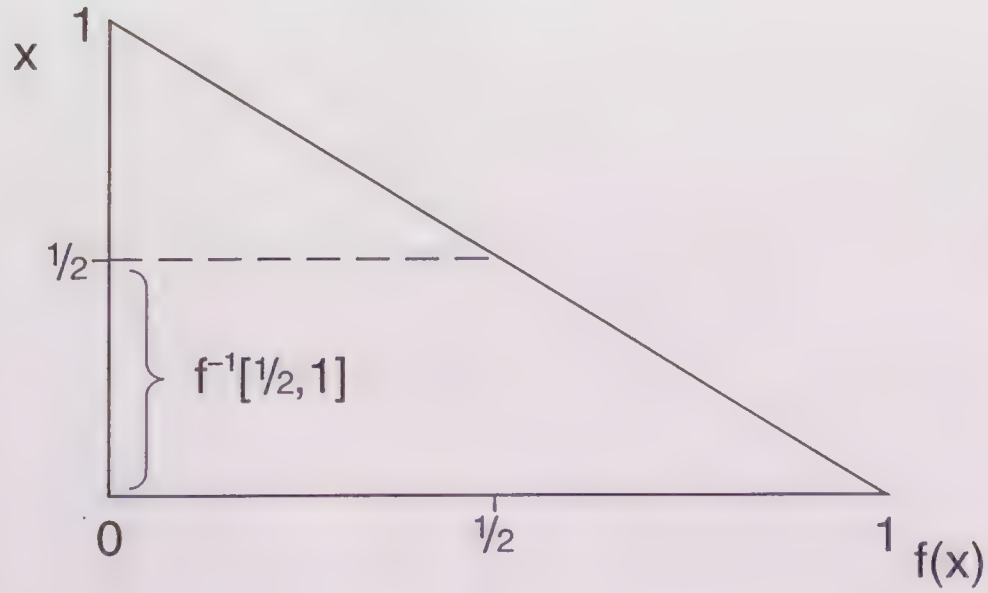


Fig 1.



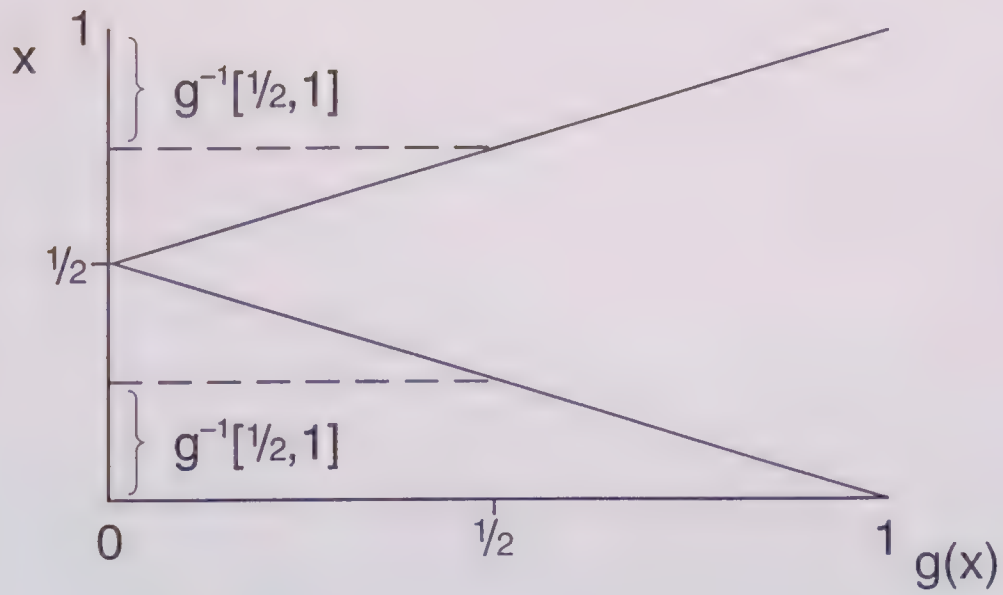
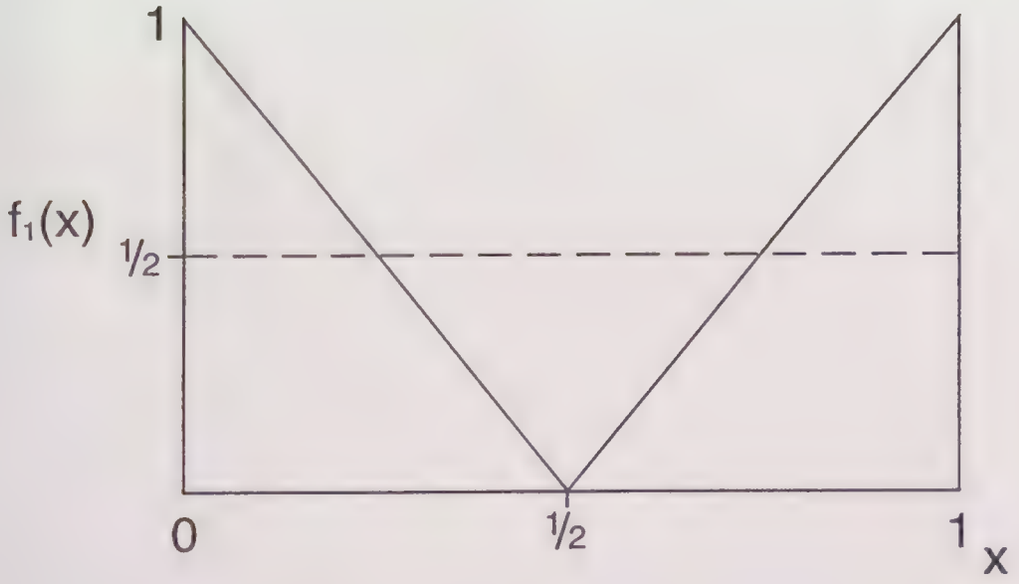


Fig.2



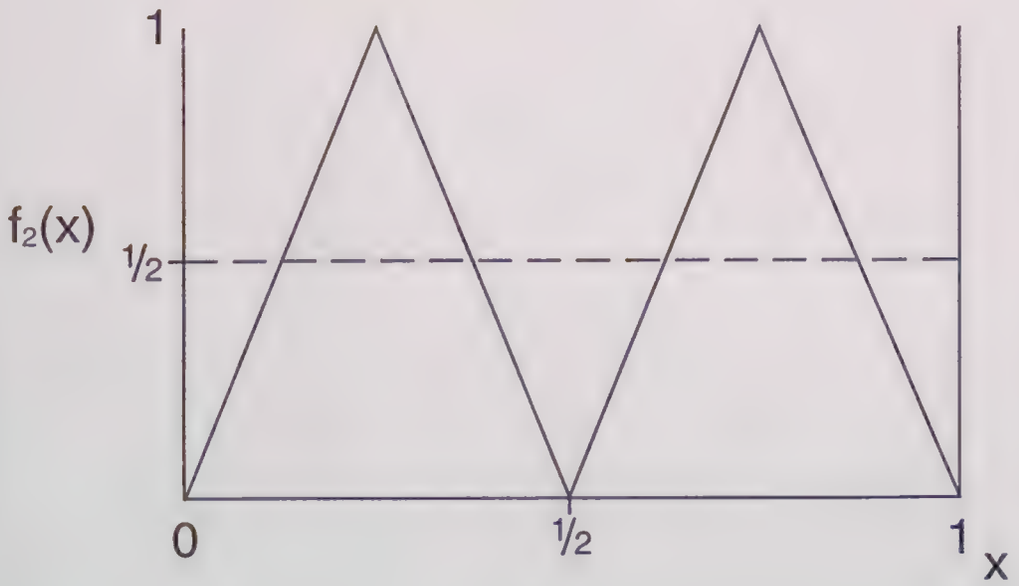
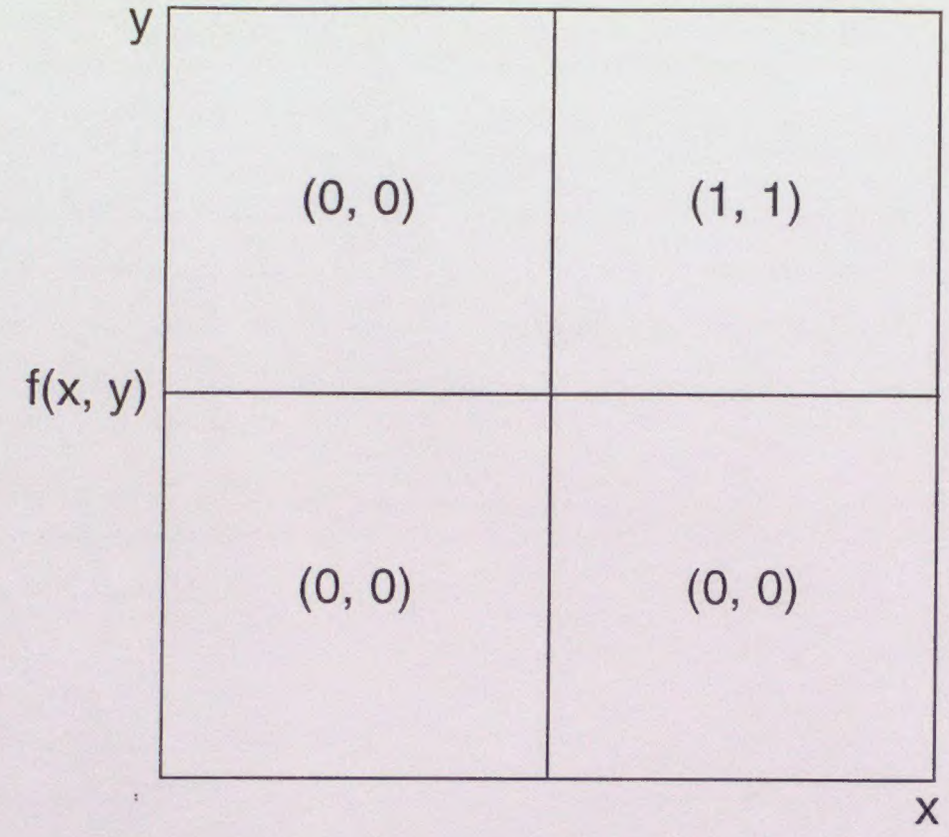


Fig.3



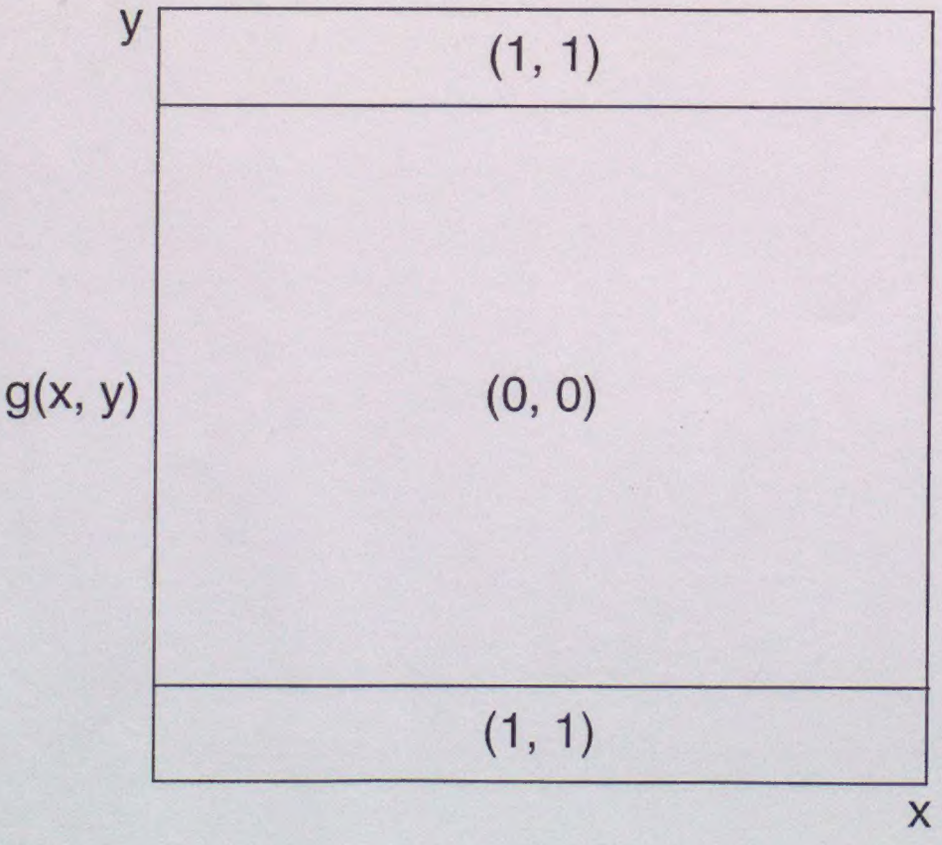
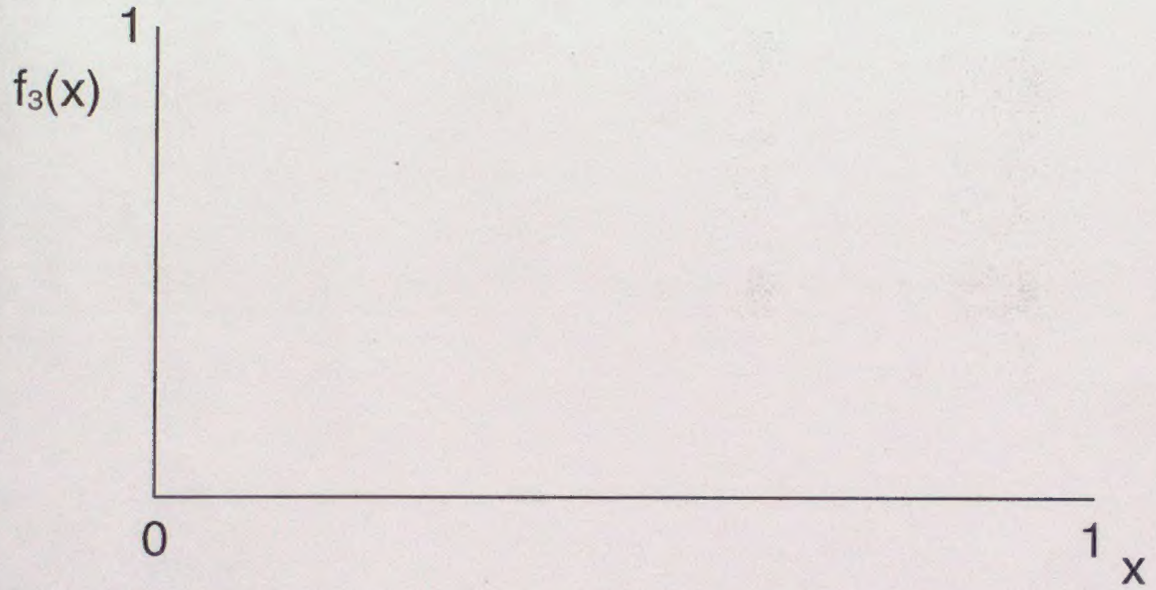


Fig.4



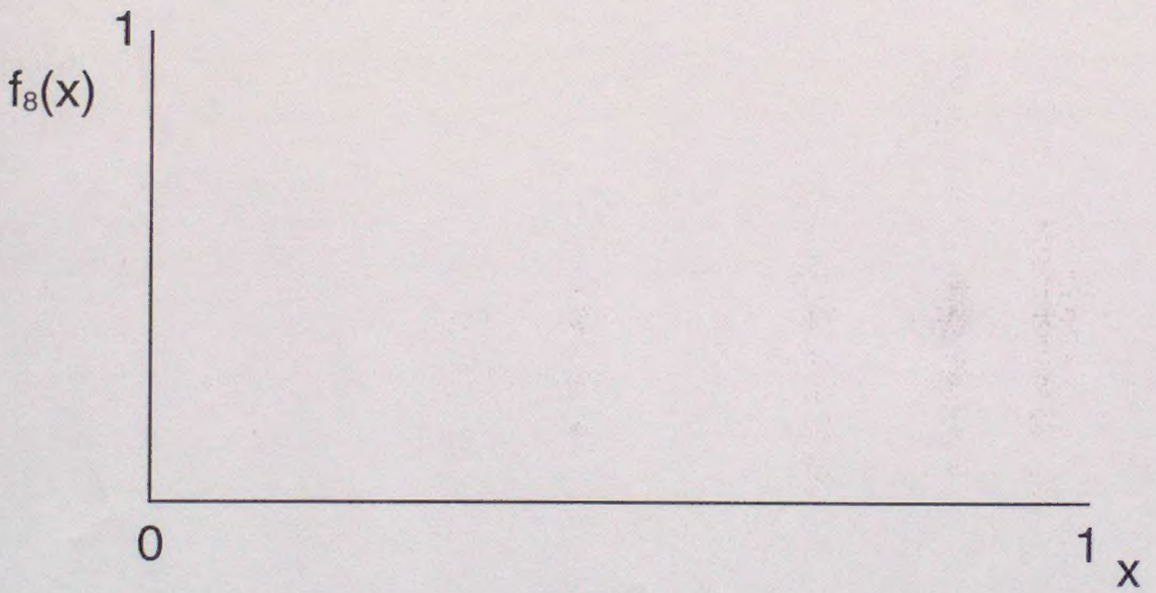


Fig.5

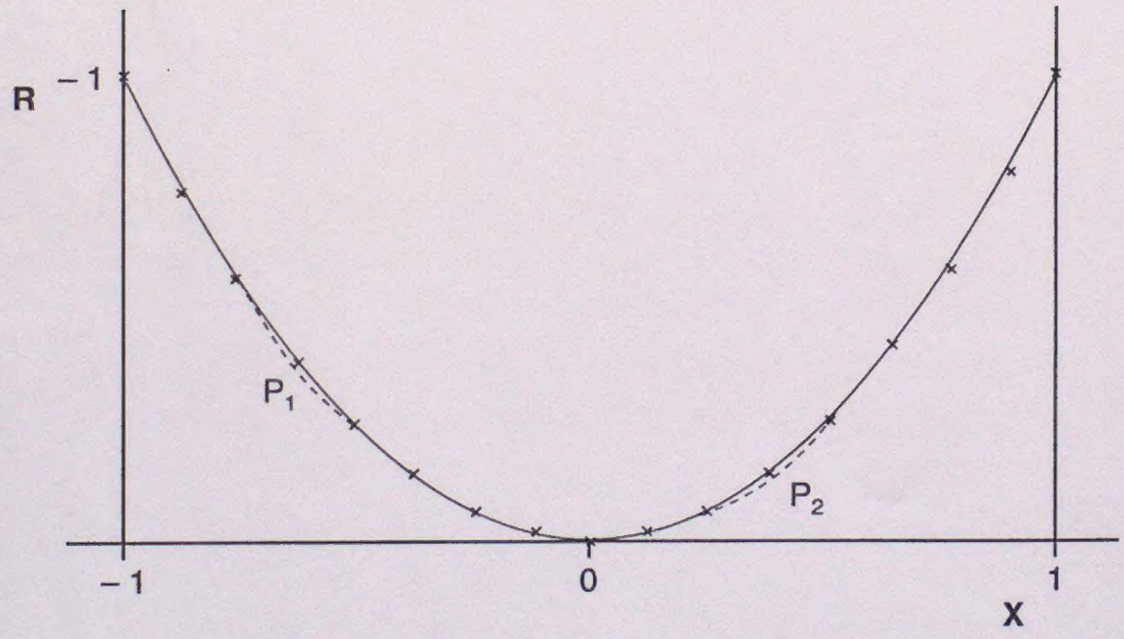


Fig.6

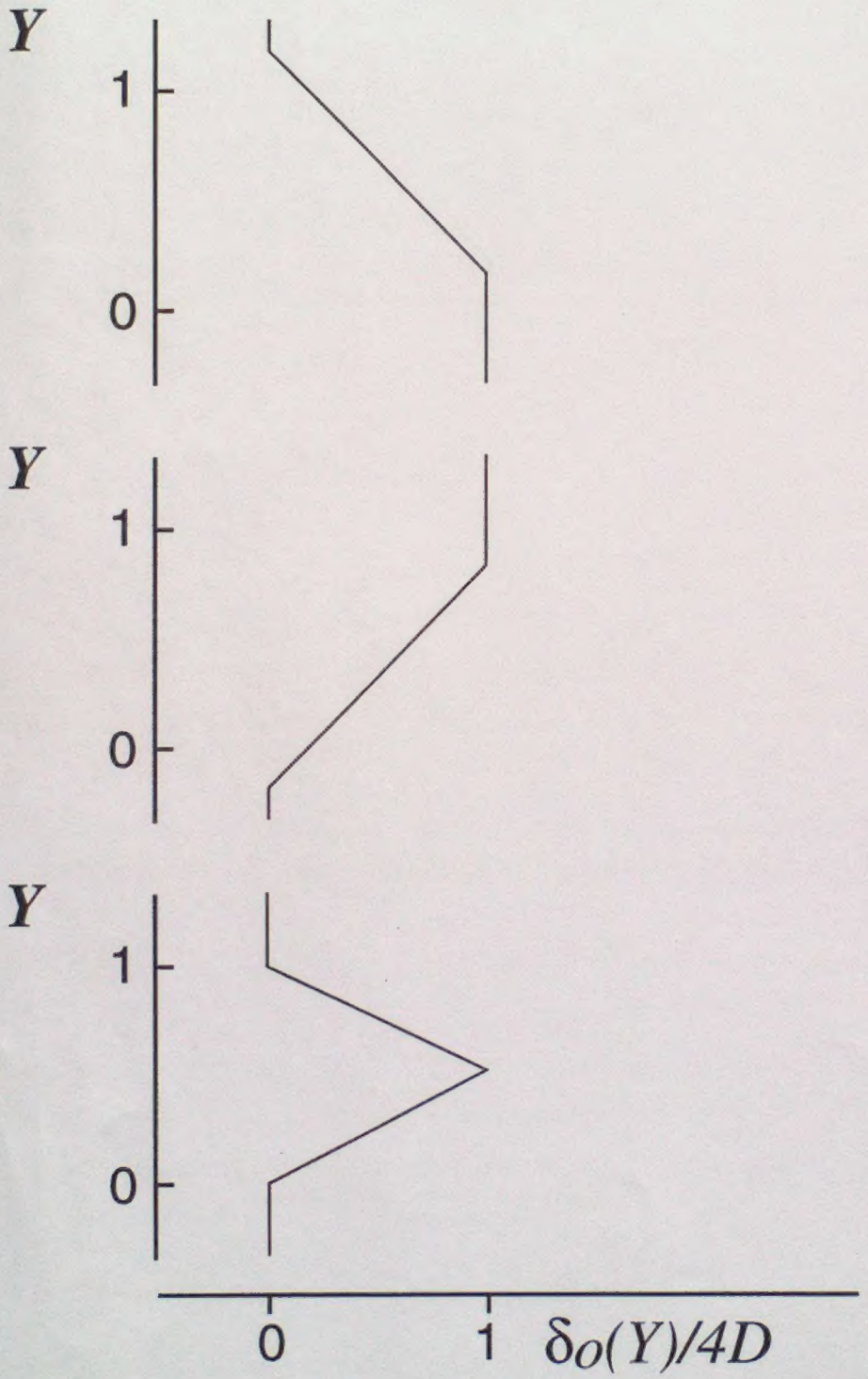


Fig 7.